

Bianchi IX Chaoticity: BKL Map and Continuous Flow

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Abstract

We analyze the Bianchi IX dynamics (Mixmaster) in view of its stochastic properties; in the present paper we address either the original approach due to Belinski, Khalatnikov and Lifshitz (BKL) as well as a Hamiltonian one relying on the Arnowitt–Deser–Misner (ADM) reduction.

We compare these two frameworks and show how the BKL map is related to the geodesic flow associated with the ADM dynamics. In particular, the link existing between the *anisotropy parameters* and the *Kasner indices* is outlined.

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The Bianchi classification characterizes all the admissible three-dimensional homogeneous spaces and determines 9 non-equivalent types. The Bianchi models were widely studied in cosmology as the simplest generalization of the Friedmann–Lemaître–Robertson–Walker Universe, the Einstein equations reducing to an ordinary differential system with respect to time [1].

Among the Bianchi classification, the types VIII and IX (Mixmaster) were particularly attractive from a cosmological point of view, being the most general ones allowed within the homogeneity constraint and characterized by a chaotic dynamics near the Big Bang; here we discuss the type IX in some detail, yet all considerations hold also for the type VIII.

The Bianchi IX model is summarized by the line element

$$ds^2 = N^2(t)dt^2 - \gamma_{ij}\sigma^i\sigma^j, \quad i, j = 1, 2, 3 \quad (1)$$

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where N denotes the lapse function and the σ^i are the 1-forms defined by the homogeneity property. In vacuum we can take

$$\gamma_{ij}(t) = \text{diag} \left(e^{h_i(t)} \right) \quad (2)$$

and the Einstein equations provide

$$h_i'' = \left(e^{h_j} - e^{h_k} \right)^2 - e^{2h_i}, \quad i \neq j \neq k \quad (3a)$$

$$\sum_i h_i'' = \frac{1}{2} \sum_{j \neq k} h_j' h_k' \quad (3b)$$

where $()' = d/d\eta$ and the time η corresponds to the choice $N = \exp \left(\sum_i \frac{h_i}{2} \right)$; equation (3b) plays the role of a first integral for the system (3a).

Belinski, Khalatnikov and Lifshitz (BKL) [1] showed that the asymptotic approach to the Big Bang of the type IX model is characterized by an infinite sequence of *Kasner epochs*, i.e. intervals of time during which the right-hand side of (3a) is negligible. Thus, in the limit $\eta \rightarrow -\infty$, we have a piecewise solution, whose single steps take the Kasner form

$$h_i \sim p_i \eta + h_i^*, \quad (p_i, h_i^*) = \text{const.}, \quad (4)$$

where the Kasner indices p_i satisfy the conditions $\sum_i p_i = \sum_i (p_i)^2 = 1$. The Mixmaster dynamics is then reduced to a map on these indices: adopting the parametrization which orders the p_i 's as

$$p_1 = -\frac{u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}, \quad u \in (1, \infty) \quad (5)$$

then we get the following discrete map on the parameter u

$$u' = \begin{cases} u-1 & u > 2 \\ \frac{1}{u-1} & 1 < u \leq 2 \end{cases} \quad (6)$$

The main achievement of the BKL approach is the discovery that such a map has stochastic properties and u admits the probability distribution w

$$w(u) = \frac{1}{u(u+1)\ln 2}. \quad (7)$$

In [2] the above analysis was extended, point by point in space, to a generic inhomogeneous cosmological model.

To get a continuous description, let us introduce the Misner variables $(\alpha, \beta_+, \beta_-)$

and then the Misner–Chitre-like ones (τ, ξ, θ) , via the transformations

$$h_1 = \beta_+ + \sqrt{3}\beta_-, \quad h_2 = \beta_+ - \sqrt{3}\beta_-, \quad h_3 = -2\beta_+ \quad (8a)$$

$$\alpha = -e^\tau \xi, \quad \beta_\pm = e^\tau \sqrt{\xi^2 - 1} \times \begin{cases} \cos \theta \\ \sin \theta \end{cases} \quad (8b)$$

with $\xi \in (1, \infty)$ and $\theta \in [0, 2\pi)$.

By an Arnowitt–Deser–Misner (ADM) reduction of the Einstein–Hilbert action, it can be shown [6] that the Bianchi IX dynamics is summarized by

$$S_{\text{IX}} = \int d\tau \left(p_\xi \frac{d\xi}{d\tau} + p_\theta \frac{d\theta}{d\tau} - \sqrt{\varepsilon^2 + U} \right), \quad (9)$$

where $\varepsilon^2 = (\xi^2 - 1)p_\xi^2 + \frac{p_\theta^2}{\xi^2 - 1}$.

Near the Big Bang ($\tau \rightarrow \infty$), the potential is modelled by the infinite walls

$$U = \sum_i \Theta_\infty(H_i), \quad \Theta_\infty(x) = \begin{cases} 0 & x > 0 \\ \infty & x \leq 0 \end{cases} \quad (10)$$

where the quantities $H_i \equiv h_i/(\sum_k h_k)$, $(\sum_i H_i = 1)$ are the *anisotropy parameters* which in the Misner–Chitre-like variables read as

$$H_1 = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi} (\cos \theta + \sqrt{3} \sin \theta) \quad (11a)$$

$$H_2 = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi} (\cos \theta - \sqrt{3} \sin \theta) \quad (11b)$$

$$H_3 = \frac{1}{3} + 2 \frac{\sqrt{\xi^2 - 1}}{3\xi} \cos \theta; \quad (11c)$$

the functions H_i expressed in (11) do not depend on the (time) variable τ . The domain Γ_H where U vanishes is dynamically closed and, within it, ε behaves as a constant of motion, i.e. $\frac{d\varepsilon}{d\tau} = \frac{\partial \varepsilon}{\partial \tau} = 0$, which implies $\varepsilon = E = \text{const.}$

Thus the Bianchi IX dynamics is isomorphic to a billiard on a Lobachevski plane whose line element reduces to

$$dl^2 = E^2 \left[\frac{d\xi^2}{\xi^2 - 1} + (\xi^2 - 1)d\theta^2 \right]. \quad (12)$$

This system is stochastic [4,6] and, upon the transformation of the conjugate momenta as

$$p_\xi = \varepsilon \frac{\cos \phi}{\sqrt{\xi^2 - 1}}, \quad p_\theta = \varepsilon \sqrt{\xi^2 - 1} \sin \phi, \quad 0 \leq \phi < 2\pi, \quad (13)$$

the dynamics admits a uniform *invariant measure* like [3,5]

$$d\mu = \frac{1}{8\pi^2} d\xi d\theta d\phi. \quad (14)$$

The link between this approach based on continuous variables and the one based on the discrete BKL one can be achieved considering the following set of coordinates transformations

$$\vec{y} = \frac{1+\xi}{\sqrt{\xi^2-1}} (\cos \theta, \sin \theta), \quad (15a)$$

$$\vec{t} = \frac{\vec{y} + \vec{b}}{(\vec{y} + \vec{b})^2} - \vec{b}, \quad (15b)$$

where \vec{t} denotes the Poincaré variables and \vec{b} belongs to the boundary of the Lobachevsky plane, i.e. $b^2 = 1$; let us consider afterwards the transformation to (u, v)

$$\vec{t} = \frac{2}{\sqrt{3}} \left[\left(u + \frac{1}{2} \right) \vec{b}^\perp + v \vec{b} \right], \quad (16)$$

$$v \geq 0, \quad -\infty < u < \infty, \quad \vec{b} = (0, 1), \quad \vec{b}^\perp = (1, 0).$$

In terms of the (u, v) variables, the anisotropy parameters (11) read

$$Q_1(u, v) = -\frac{u}{\delta}, \quad Q_2(u, v) = \frac{1+u}{\delta}, \quad Q_3(u, v) = \frac{u(u+1)+v^2}{\delta}, \quad (17)$$

with $\delta(u, v) \equiv (u^2 + u + 1 + v^2)$. The quantities in (17) reduce to the Kasner exponents (5) when evaluated at $v = 0$.

This correspondence provides the link between the dynamics representation in continuous variables and the BKL map [5].

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